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On interactions between imperfections in cellular solids

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A number of studies have indicated that imperfections in cellular solids can have a very large influence on their mechanical behavior. Waviness in cell walls can drastically reduce elastic stiffnesses, and various irregularities can reduce the effective strengths by large factors. These imperfections have in many instances been studied in isolation, i.e., only a singe type of imperfection was studied at a time. The goal of the present paper is to shed light on how different kinds of imperfections interact. There is a potential risk that when multiple kinds of imperfections are present, the mechanical properties of the cellular solid are worse than what would be predicted from each one in isolation. The present study is focused on linear elastic properties, although many of the conclusions apply also for other properties. The method employed is based on statistical averages and Taylor expansion arguments. Such techniques have in previous papers been applied to a single imperfection type. The conclusion from the present analysis is that, up to the third order in "small" parameters describing the severity of the imperfections, there is no interaction between a number of different types of imperfections.

1. Introduction

There exist a large number of idealized models for "perfect" (3D) closed cell cellular solids (e.g., Gibson and Ashby [1], Grenestedt [2]). Some of these models predict properties of, for example, expanded PVC foams fairly well. However, these models tend to overestimate the properties of some other cellular solids, such as aluminum foams (e.g., [3]). The reason is believed to be attributed to various kinds of geometric "imperfections" in the cellular solids, or deviations from the idealized models. A number of different kinds of imperfections in (3D) cellular solids have been studied, including cell wall waviness (e.g., Grenestedt [4], Simone and Gibson [5]), cell shape variations (e.g., Grenestedt and Tanaka [6]), cell wall thickness variations (e.g., Grenestedt and Bassinet [7]), cell wall removal, etc. These imperfections have usually been studied in isolation, i.e., with only one imperfection present in the cellular solid. However, in real cellular solids there are in general many different kinds of imperfections present, as seen in Fig. 1. The present paper presents some presumably new results for the case when multiple imperfections are present. The results are based on statistical arguments similar to those used by Grenestedt and Bassinet [7]. The paper is arranged as follows: a particular "perfect" model that has been used frequently as a reference model for studies on imperfections, the Kelvin foam with flat cell walls, is first presented. Some of the results from previous investigations on the influence of imperfections are then summarized. The subsequent section deals with the interaction of imperfections and the paper ends with a summary and some conclusions.

2. Perfectly ordered Kelvin model

The perfectly ordered Kelvin foam consists of polyhedra with 14 flat faces (tetrakaidecahedra) as depicted in Fig. 2. These polyhedra pack in a body centered cubic (BCC) arrangement, and completely fill space. Kelvin (Thomson [8]) used this model, but with slightly curved faces, to model liquid foams. All polyhedra have the same size, and are cubically symmetric. The stiffness of the resulting cellular solid has cubic, but not isotropic, symmetry. For cellular solids which are not isotropic, we will present isotropic stiffness obtained by averaging stiffnesses over all directions; for a full discussion, see Kröner [9] and Grenestedt [2].

The results from Grenestedt's [2] Finite Element (FE) analyses show that Young's modulus and bulk modulus of this foam scale virtually linearly with density. The bulk modulus is very close to the theoretical maximum limit, provided by the Hashin-Shtrikman

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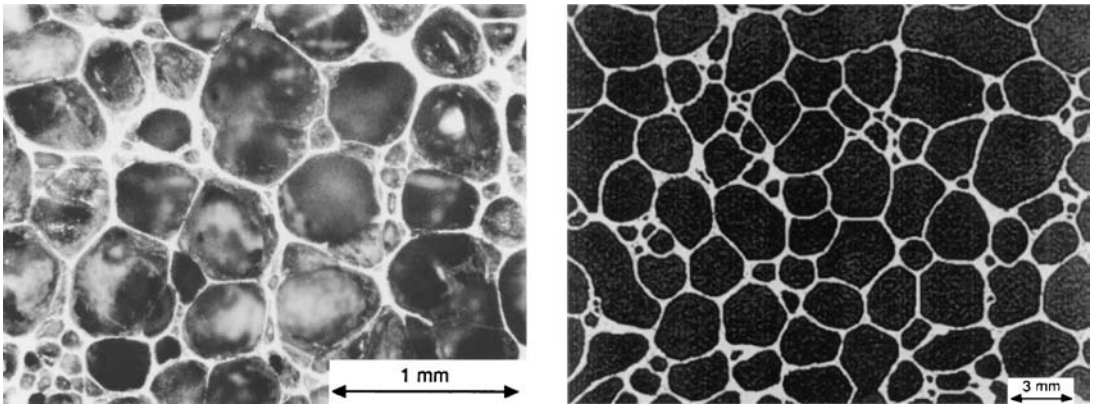

Figure 1 Expanded PVC foam (left) and aluminum foam (right). The PVC foam has better relative properties than the aluminum foam, a fact which is believed to be due mainly to less curvature in the cell walls. However, many kinds of imperfections are present, including wavy cell walls, non-uniform cell wall thicknesses, different shapes and sizes of cells, etc.

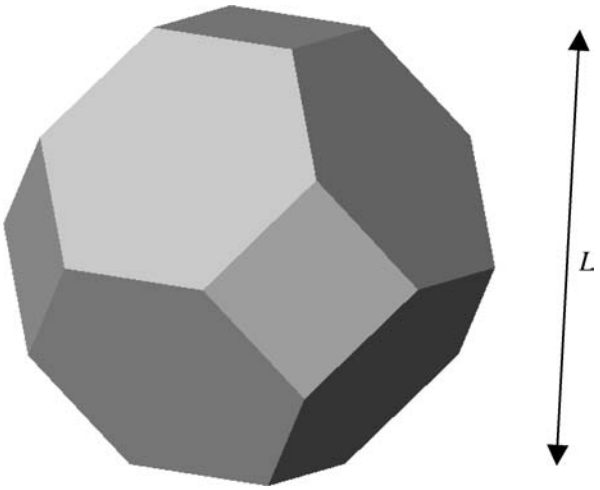

Figure 2 A 14-sided cell in a flat faced Kelvin foam. The "diameter" of the cell is L.

[10] upper bound,

$$\bar{K}^{\mathrm{HS}} = \frac{4GK(\bar{\rho}/\rho)}{4G + 3K(1-\bar{\rho}/\rho)} \underset{\frac{\bar{\rho}}{\rho}\to 0}{\rightarrow} \frac{\bar{\rho}}{\rho}\frac{4GK}{4G+3K} = \frac{\bar{\rho}}{\rho}\frac{2E}{9(1-\nu)} \tag{1}$$

The low density asymptote is of particular interest for the presently considered low density cellular solids. The Young's modulus is for the flat-faced Kelvin cellular solid

$$\frac{\bar{E}}{E} \approx 0.35\frac{\bar{\rho}}{\rho} \tag{2}$$

when Poisson's ratio for the solid is $\nu = 0.3$. The low density asymptote of the Hashin-Shtrikman upper bound on Young's modulus is

$$\bar{E}^{\mathrm{HS}} = \frac{9\bar{K}\bar{G}}{3\bar{K}+\bar{G}} = \frac{\bar{\rho}}{\rho}\frac{2(7-5\nu)E}{3(1-\nu)(9+5\nu)} \tag{3}$$

For $\nu = 0.3$, this becomes $\bar{E}^{\mathrm{HS}} \approx 0.50E(\bar{\rho}/\rho)$. The reference model thus has almost the highest possible bulk modulus, and a Young's modulus which is 30% lower than the maximum possible Young's modulus.

The elastic properties which this model predicts quite well match the experimentally measured properties of expanded PVC (Divinycell [11–13]) of three different grades and with densities ranging from 36 to 400 kg/m^3, see Grenestedt [2]. The good correlation with experiments was the main reason for selecting this model as a reference model. However, it overestimates the stiffnesses of aluminum foams.

3. Wavy distortions of cell walls

In this Section, wavy distortions of the cell walls are discussed. For sake of simplicity, the cell walls will be assumed to have constant thickness. Details of the derivations are given by Grenestedt [4] but are presently omitted.

In order to get some insight into the mechanics of this kind of imperfection, a rod is first studied. If the rod is straight and stretched by a force F, then the elongation δ is

$$\delta = \frac{Fa}{EA} \tag{4}$$

where E is Young's modulus of the material in the rod, A is the cross sectional area of the rod, and a is the length of the rod. If the rod has a shallow initial wavy distortion with the shape

$$\hat{w}_0\left(1 - \cos\frac{2m\pi x}{a}\right) \tag{5}$$

where $\hat{w}_0$ is the amplitude of the wavy imperfection, x is the coordinate along the length of the rod, and m is an integer, then the force-elongation relationship for small elongations becomes

$$\delta = \frac{Fa}{EA}\left(1 + \frac{\hat{w}_0^2 A}{2I}\right) \tag{6}$$

where I is the area moment of inertia of the rod. A number of details have been left out here, but these have little consequence for the present discussion. Assume

for the sake of simplicity that the rod has a rectangular cross section with the side h. The stiffness of the rod then becomes

$$\frac{F}{\delta} = \frac{Eh^2}{a\left(1 + 6\left(\frac{\hat{w}_0}{h}\right)^2\right)} \tag{7}$$

and this provides good insight into the influence of wavy imperfections on stiffnesses. If the initial wavy imperfection has an amplitude equal to the thickness of the rod, $\hat{w}_0 = h$, then the stiffness is a seventh of the stiffness of the straight rod, and with $\hat{w}_0 = 2h$ the stiffness is only 4%.

We now turn to closed cell 3D models. The only geometry considered in the present model is the shape of a single cell wall, which in some sense is representative for the whole cellular solid. All cell walls in the model are assumed to be made of the same material and have the same shape, but they may have different sizes. This simplification allows upper bounds to be derived using the single cell wall. The shape of a representative cell wall naturally depends on the foam. Grenestedt [4] selected a square plate with side a, uniform thickness t, and with the wavy imperfections

$$w_0(x, y) = \hat{w}_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \tag{8}$$

as a representative cell wall shape. A square shaped cell wall is not substantially more unphysical than a pentagon, hexagon or any other similarly shaped wall. The square plate is simple to model and provides the desired results. The goal of the study was not to get a highly accurate value of actual stiffnesses, but to provide insight into some of the mechanisms. Stiffnesses were calculated using the Finite Element Method (FEM). Here, x and y are Cartesian coordinates, $\hat{w}_0$ is the amplitude of the waviness, and m and n integers. Details are provided in Grenestedt [4] and only some results are presented here. The shear and bulk moduli for the cases $m = n = 1$ and $m = n = 2$ are given in Fig. 3.

For vanishing waviness amplitude, $\hat{w}_0 = 0$, the present 3D model predicts the same stiffnesses as the low density asymptotes of the Hashin-Shtrikman upper bounds, i.e., $\bar{K}/E \approx 0.32(\bar{\rho}/\rho)$, $\bar{G}/E \approx 0.20(\bar{\rho}/\rho)$ and $\bar{E}/E \approx 0.50(\bar{\rho}/\rho)$ when $\nu = 0.3$. The present model predicts the Young's modulus $(\bar{E}/E)/(\bar{\rho}/\rho) =$

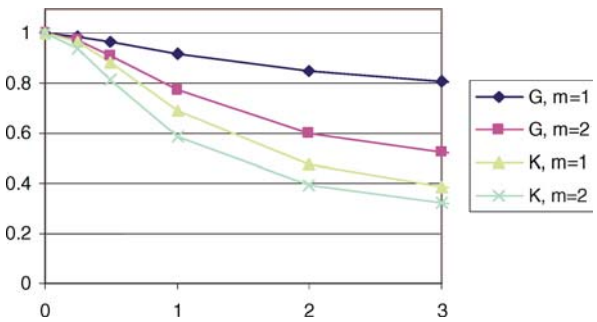

Figure 3 Normalized shear (G) and bulk (K) moduli plotted versus waviness amplitude $\hat{w}_0/t$ where t is the thickness of the cell walls. The moduli are normalized by the values they assume with no imperfections.

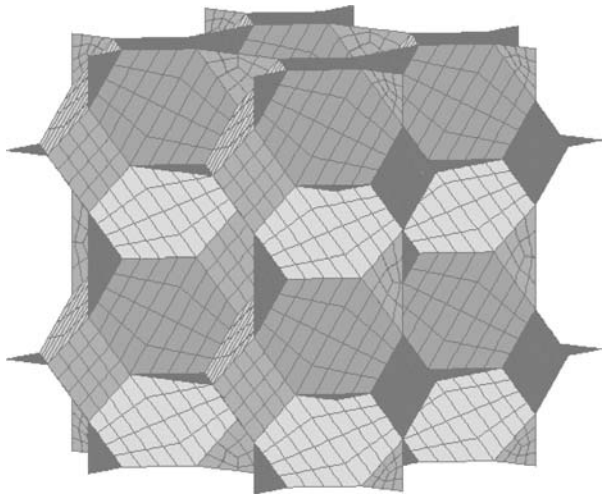

Figure 4 Kelvin foam model with a total volume equal to that of 16 cells.

0.31 for $\hat{w}_0/t = 5$, $m = n = 1$, and $t/a = 0.03$, which is a 38% decrease relative to the foam with no cell wall waviness, and $(\bar{E}/E)/(\bar{\rho}/\rho) = 0.28$ for $\hat{w}_0/t = 2$, $m = n = 2$, and $t/a = 0.003$, which is a 44% decrease. These parameters appear to be fair estimates of the wavy imperfections in the Alporas aluminum foam, seen in Fig. 1 (right).

4. Cell wall thickness variations

The influence of cell wall thickness variations has been studied by Grenestedt and Bassinet [7], and some of the results are discussed here. The Kelvin model presented previously was used as a reference, and cell wall thickness variations were introduced in this model. The resulting stiffnesses have no bounding properties.

A model as depicted in Fig. 4, which has the volume of 16 of the polyhedra introduced in Fig. 2, was chosen as a unit cell for all analyses. This model was then repeated periodically in three dimensions. Rigorous periodicity boundary conditions were applied and all stiffnesses calculated using FEM.

There are 112 different cell walls in the unit cell. The 112 walls were allocated thicknesses randomly, as

$$t_i = t_0(1 + \alpha\varphi_i)c, \quad i = 1, 2, \ldots, 112 \tag{9}$$

where the amplitude of the cell wall thickness variation $\alpha \in (0, 1)$ is a real number, $\varphi_i \in (-1, 1)$ is a random number with a uniform distribution, and t_0 is the cell wall thickness of the perfect structure, i.e. when $\alpha = 0$. A random distribution is likely to change the density of the cellular solid (the amount of material within the unit cell), and therefore all cell walls were multiplied with the normalizing factor c to keep the density of the cellular solid constant.

Details of the analysis are omitted, but the results in terms of the variation of shear and bulk moduli with thickness imperfection amplitude α are presented in Fig. 5. The relative density of the foam in the figure is $\bar{\rho}/\rho = 0.01$. As seen in the figure, the moduli are not very sensitive to cell wall thickness variations in this model. Even for $\alpha = 0.9$, which means that the thickest

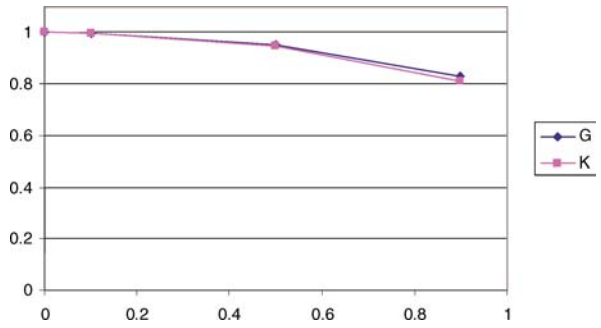

Figure 5 Normalized shear (G) and bulk (K) moduli plotted versus the amplitude of the cell wall thickness variation α.

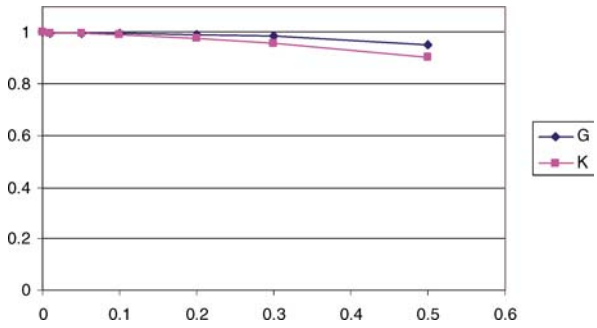

Figure 6 Normalized shear (G) and bulk (K) moduli plotted versus the amplitude of the cell shape non-uniformity parameter β.

cell wall can be 19 times as thick as the thinnest, the moduli not even decreased by 20%.

5. Cell shape variations

The last imperfection discussed in some depth in this paper is cell shape variations, as studied by Grenestedt and Tanaka [6]. The larger Kelvin model of Fig. 4 was again used as a reference. The cell walls of the reference model coincide with the boundaries of the Voronoi sets[1] made from points, or "kernels", arranged in a BCC fashion. The non-uniform models were created by moving the kernels from the original positions of the perfect Kelvin foam in a random fashion. However, movements of the kernels were limited to be no more than βL, where β is an amplitude of the cell shape non-uniformity and L is the "diameter" of the perfect Kelvin polyhedron; see Fig. 2. The coordinates of the kernels of the perfect Kelvin foam, x_i^k, were thus perturbed as follows:

$$x_i^k = \tilde{x}_i^k + \beta L \varphi_i^k \tag{10}$$

where the amplitude of the cell shape non-uniformity β is a real number and $\varphi_i \in (-1, 1)$ is a stochastic variable with a uniform distribution. Voronoi sets were made from these kernels, and thus the non-uniform cell models were obtained. Just as in the last Section, periodicity was assumed with 16 polyhedra in the unit cell (Fig. 4), and cell wall thicknesses were adjusted to keep the effective density of the foam models constant.

We again omit the details of the FE analysis, but present the variation of shear and bulk moduli with amplitude of the cell shape non-uniformity β in Fig. 6. The relative density of the foam in the figure is $\bar{\rho}/\rho = 0.01$. As seen in the figure, the moduli are not very sensitive to cell shape variations in this model. Even for $\beta = 0.5$, the shear modulus has only decreased by less than 10%.

6. Interactions between imperfections in cellular solids

Assume that there are two independent imperfections present in the cellular solid, and that the severities of these imperfections are scaled by the parameters α and β. The effective properties of the foam (e.g., shear modulus G) will then depend on α and β. If the imperfections are "small", G may be expanded in a Taylor series:

$$G = G_0 \sum_{\substack{i=0,1,2,\ldots \\ j=0,1,2,\ldots}} k_{ij} \alpha^i \beta^j \tag{11}$$

where $k_{00} = 1$. For small imperfections, the higher order terms may be neglected. Neglecting fifth order terms and higher, the following approximation is obtained

$$\begin{aligned} G = G_0(1 &+ k_{01}\beta + k_{02}\beta^2 + k_{03}\beta^3 + k_{04}\beta^4 + k_{10}\alpha \\ &+ k_{11}\alpha\beta + k_{12}\alpha\beta^2 + k_{13}\alpha\beta^3 + k_{20}\alpha^2 + k_{21}\alpha^2\beta \\ &+ k_{22}\alpha^2\beta^2 + k_{30}\alpha^3 + k_{31}\alpha^3\beta + k_{40}\alpha^4 + O(\alpha, \beta)^5) \end{aligned} \tag{12}$$

The parameters α and β are for many types of imperfections not limited to non-negative numbers. For example, a negative $\hat{w}_0$ makes complete sense in Equations 5 and 8, as do a negative α in Equation 9 and a negative β in Equation 10. Due to the statistical nature, $G(\alpha,\beta) = G(-\alpha,\beta) = G(\alpha,-\beta)$. This requires the following constants in the Taylor expansion to vanish: k_{01}, k_{03}, k_{10}, k_{11}, k_{12}, k_{13}, k_{21}, k_{30}, k_{31}, etc. Thus, to the fourth order,

$$\begin{aligned} G = G_0(1 &+ k_{20}\alpha^2 + k_{02}\beta^2 + k_{40}\alpha^4 \\ &+ k_{04}\beta^4 + k_{22}\alpha^2\beta^2) \end{aligned} \tag{13}$$

In conclusion, to the third order there is no interaction between imperfections (the lowest order term including interactions is $k_{22}\alpha^2\beta^2$). Extending the argument to an arbitrary number of imperfections, as well as to other properties of the cellular solid, is straight forward. The argument requires that the parameters describing the imperfections may take both positive and negative values without changing the cellular solids's properties in a statistical sense.

Li *et al.* [14] recently calculated elastic properties of 2D cellular solids with two kinds of imperfections simultaneously present. Their numerical results also showed that there was very little interaction between the two kinds of imperfections in their materials.

[1]The Voronoi set V^k of kernel p^k is the set of points in space which are closer to p^k than to any other kernel p^l, where $k \neq l$. Each Voronoi set V^k (which here is a volume) is convex and encloses only one kernel, p^k. The boundaries of the Voronoi sets are flat.

7. Conclusions

In this paper, the influences of some different kinds of imperfections in cellular solids were revisited. These imperfections were waviness of cell walls, thickness variations of cell walls, and shape variations of cells. More recent work is presented by Ribeiro-Ayeh and Hallström in the Thesis of the former [15]. The results from these studies suggest that stiffnesses are most sensitive to cell wall waviness. Further, using Taylor expansion and statistical arguments, it was shown that interactions between imperfections are very weak for small imperfections. This means that if, say, two different kinds of imperfections are present in a cellular solid, then to the third order their influence on stiffness reductions are simply additive. For example, if a certain cell wall waviness reduces the stiffness by 5% and a certain cell thickness variation reduces stiffness by 3%, then having both imperfections present simultaneously in a cellular would reduce the stiffness by approximately 8%.

Acknowledgments

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References

1. L. J. GIBSON and M. F. ASHBY, "Cellular Solids: Structures and Properties", 2nd edition (Cambridge University Press, Cambridge, 1997).
2. J. L. GRENESTEDT, *Int. J. Solids Struct.* **36**(10) (1999) 1471.
3. Y. SUGIMURA, J. MEYER, M. Y. HE, H. BART-SMITH, J. GRENESTEDT and A. G. EVANS, *Acta Mater.* **45** (12) (1997) 5245.
4. J. L. GRENESTEDT, *J. Mech. Phys. Solids* **46**(1) (1998) 29.
5. A. E. SIMONE and L. J. GIBSON, *Acta Mater.* **46**(11) (1998) 3929.
6. J. L. GRENESTEDT and K. TANAKA, *Scripta Materialia* **40**(1) (1998) 71.
7. J. L. GRENESTEDT and F. BASSINET, *Intern. J. Mech. Sci.* **42**(7) (2000) 1327.
8. W. THOMSON (LORD KELVIN), *Phil. Mag.* **24**(151) (1887) 503.
9. E. KRÖNER, *Zeitschrift für Physic* **151** (1958) 504.
10. Z. HASHIN and S. SHTRIKMAN, *J. Mech. Phys. Solids* **11** (1963) 127.
11. "Divinycell, Technical Manual HT Grade" (DIAB, Box 201, S-312 22 Laholm, Sweden, 1992).
12. "Divinycell, Technical Manual HCP Grade" (DIAB, Box 201, S-312 22 Laholm, Sweden, 1992).
13. "Divinycell, Technical Manual H Grade" (DIAB, Box 201, S-312 22 Laholm, Sweden, 1995).
14. K. LI, X.-L. GAO and G. SUBHASH, *Intern. J. Solids Struct.* **42** (2005) 1777.
15. S. RIBEIRO-AYEH, Finite Element Modelling of the Mechanics of Solid Foam Materials, PhD. Thesis, ISBN 91-7178-002-5, Royal Institute of Technology, Dept. of Aeronautical and Vehicle Eng., S-100 44 Stockholm, Sweden, 2005.

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